Smooth morphisms

Definition 1 *A morphism $f\colon X \to Y$ is smooth if it is locally of finite presentation and formally smooth.*

It is clear from the definition that I is an ideal in an R-algebra A, then the map $\mathbf{A}^n(A) \to \mathbf{A}^n(A/I)$ is surjective. This is in particular true if I is nilpotent so that $\mathbf{A}^n \to \operatorname{Spec} R$ is formally smooth, hence smooth.

Theorem 2 *Let $f\colon Z \to Y$ be a smooth morphism of schemes, let $i\colon X \to Z$ be a closed immersion defined by a sheaf of ideals I, which we assume to be of finite type. Then the following are equivalent:*

1. *X/Y is smooth.*
2. *The map $I/I^2 \to i^*\Omega_{Z/Y}$ is injective and locally split.*

Proof: Let $T \to T'$ be an first order thickening of affine Y-schemes, with ideal I_T, $h\colon T \to X$ be a morphism. Since Z/Y is smooth, h can be deformed to a map $h'\colon T \to Z$. We need h' to factor through X, *i.e.*, we need

$$h'^*\colon I_X \to h_* I_T$$

to be zero. Since $I_T^2 = 0$, this map factors through I_X/I_X^2. Splitting

$$\sigma\colon i^*\Omega_{Z/Y} \to I_X/I_X^2$$

composed with h'' gives

$$\Omega_{Z/Y} \to h_* I_T.$$

Use this to change the deformation h' to a new one which works.

For the converse, look at the first infinitesimal nbd. X_1 of X in Z. Smoothness gives a deformation $X_1 \to X$, and we use this to get a section as before. □

Corollary 3 *If X/Y is smooth, $\Omega_{X/Y}$ is locally free.*

Corollary 4 *Let Z/Y be a smooth morphism and let $i\colon X \to Z$ be a closed immersion with ideal I, and let x be a point of X. Then the following are equivalent:*

1. *There is an open neighborhood U of x which is smooth over Y.*
2. *The map $I(x) \to \Omega_{Z/Y}(x)$ induced by d is injective.*

Proof: Suppose (2) holds. Choose a basis for the k-vector space I/mI and lift it to a sequence $(a_1, \ldots, a_r)$ in I_x. By Nakayama's lemma, this sequence generates I_x. By hypothesis, the image of $(a_1, \ldots, a_r)$ in $\Omega_{Z/k}(x)$ is linearly independent, and hence can be extended to a basis $\omega.(x)$ for $\Omega_{Z/k}(x)$. Since Z/k is smooth, $\Omega_{Z/k,x}$ is a free $\mathcal{O}_{Z,x}$-module, any lift $\omega.$ of $\omega.(x)$ to $\Omega_{Z/k,x}$ will be a basis. It follows that the map $I_x/I_x^2 \to i^*\Omega_{Z/k,x}$ is injective and locally split. The same holds in some neighborhood of x, so (1) follows from Theorem 2. The proof that (1) implies (2) is immediate from this theorem. □

Theorem 5 *Let $X \to Z \to Y$ be morphisms of schemes. Assume that X/Y and Z/Y are smooth. Then X/Z is smooth if and only if locally on X the map $g^*\Omega_{Z/Y} \to \Omega_{X/Y}$ is injective and locally split.*

Corollary 6 *Let $X \to Y$ be a smooth morphism. Then, locally on X, there exists an étale factorization $X \to \mathbf{A}^n_Y \to Y$ of X/Y.*

Proof: Let x be a point of X. The image of $\mathcal{O}_{X,x} \to \Omega_{X/Y}(x)$ generates the $k(x)$-vector space $\Omega_{X/Y}(x)$, so there exists a sequence $(a_1, \ldots, a_n)$ in $\mathcal{O}_{X,x}$ whose image is a basis for $\Omega_{X/Y}(x)$. Get map $g \colon X \to \mathbf{A}^n_Y$ with $g^*t_i = a_i$ and $g^*dt_i = da_i$. Then

$$g^*\Omega_{\mathbf{A}^n_Y/Y} \to \Omega_{X/Y}$$

is an isomorphism. By the previous result, X/Y is smooth, and since $\Omega_{X/\mathbf{A}^n d_Y} = 0$, it is also unramified. □

Example 7 *Let X be the closed subscheme of affine two space over $\mathbf{Z}[t]$ defined by $(x_1^3+x_2^3+1-3tx_1x_2)$. Compute where $X/\mathbf{Z}$ is smooth and where $X/\operatorname{Spec}\mathbf{Z}[t]$ is smooth. Do the same for the equation $t(x_1^3+x_2^3+1)-3x_1x_2$, and for $t(x_1^4+x_2^4+x_3^4+1)-4x_1x_2x_3$.*